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Percolation properties of the non-ideal gas

E. Pechersky† and A. Yambartsev‡

† IITP, 19, Bolshoj Karetny per., Moscow, Russia

pech@iitp.ru

‡ IME-USP, Rua do Matão, 1010, 05508-090, São Paulo, Brazil

yambar@ime.usp.br

Abstract

We estimate locations of the regions of the percolation and of the non-percolation in the plane (λ, β): the Poisson rate – the inverse temperature, for interacted particle systems in finite dimension Euclidean spaces. Our results about the percolation and about the non-percolation are obtained under different assumptions. The intersection of two groups of the assumptions reduces the results to two dimension Euclidean space, $\mathbb{R}^2$, and to a potential function of the interactions having a hard core.

The technics for the percolation proof is based on a contour method which is applied to a discretization of the Euclidean space. The technics for the non-percolation proof is based on the coupling of the Gibbs field with a branching process.

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1 Introduction

A rigorous proof of phase transitions for continuous models of the statistical mechanics is still an open problem if the interactions between particles are described by conventional in physics potential functions. The first and yet to this moment the only example of the rigorous proof of the phase transition in a continuous model is the result by J.L. Lebowitz, A. Mazel and E. Presutti

in [3]. The potential functions in [3] are a pre-limiting version of the mean-field interaction, see [2], having a large but finite radius of the interactions, and a four-body stabilizing potential function.

In the present work, we investigate the percolation properties of an interacting particle ensemble. We describe a phase diagram of the continuous system in the plane (λ, β), *the Poisson rate – the inverse temperature*. The interaction is defined by pair potential functions. We do not prove the phase transition driven by boundary conditions as in [3]. However we think that the transition: *the percolation – the non-percolation* can be considered as a phase transition relatively, for example, to the conductivity of the matter or the velocity of the sound propagations.

The book [5] gives a rather complete picture of the state of the continuum percolation theory for the ideal gas from the mathematical point of view. Much attention in [5] is drawn to the Boolean percolation problem for the Poisson point processes in $\mathbb{R}^\nu$. Points of a configuration of the process are considered as the centers of closed balls of a random radius (*Boolean radius*) such that the radii corresponding to different points are independent of each other (and also independent of the process) and identically distributed. The existence of an unbounded connected component in the set composed by the union of all random balls means the percolation. The unbounded component is called an infinite cluster. One of the main results in [5] which is related to the present article is about the existence of a critical value λ_c of the rate of the Poisson point processes. Namely, the value λ_c distinguishes the percolation and the non-percolation, where the last means that with the probability 1 only bounded connected components exist in the union of the balls. It is asserted in [5] that with the probability 1 there are no infinite clusters when $\lambda < \lambda_c$ and there exists an infinite cluster when $\lambda > \lambda_c$.

We consider the same problem but for a non-ideal gas, which is determined by some interaction potential function and the Poisson free measure with the rate λ. A non-percolation condition was studied in the work [10] for positive finite range potential functions. The Boolean radius is equal to the range of the interaction. A new proof of this result can be found in [11].

Next we give a brief description of our results not concerning the conditions. We study the case when the potential function takes as positive as negative values. The potential function determines a Gibbs measure of which the percolation properties we investigate. In this case a new parameter enters into the game it is temperature T. By a tradition we more often use the inverse temperature $\beta = \frac{1}{T}$. The results we present here outline two

regions in the plane (λ, β) of the percolation and the non-percolation with probability 1 for a Boolean radius ℓ. We do not seek the solution as precise as possible. Our aim is to outline the regions such that they have typical forms. Namely (see Figure 1):

The region of the non-percolation can be described as follows.

There exists a density value λ_ℓ^- such that for any $\lambda < \lambda_\ell^-$ there exists an inverse temperature $\beta_\ell^-(\lambda)$ such that for any $\beta < \beta_\ell^-(\lambda)$ all clusters are finite with Gibbs probability 1.

The region of the percolation can be described as follows.

For any density λ there exists an inverse temperature $\beta_\ell^+(\lambda)$ such that for all $\beta > \beta_\ell^+(\lambda)$ there exists an infinite cluster with Gibbs probability 1. There exists a density value λ_ℓ^+ such that $\beta_\ell^+(\lambda) = 0$ if $\lambda > \lambda_\ell^+$

Our results provide estimates of the parameter regions separating the areas of the existence A_+ and of the non-existence A_- of an infinite cluster. There exists a region between A_+ and A_- where our result does not give the answer on the percolation.

The result shows in particular that for any small density λ there exists an infinite cluster if temperature is low enough. Another feature is also that the non-existence of an infinite cluster may only be at a small density, $\lambda < \lambda_\ell^-$. This fact is in accord with the result ([5]) of the Boolean non-percolation for the ideal gas.

The conditions under which we prove the percolation and the non-percolation are different. We prove the percolation result in $\mathbb{R}^2$ only. It is necessary as well, that the potential function has an attractive part. However, we do not assume a hard core. Our proof of the non-percolation requires the hard-core condition. The attractive part of the potential as well as the dimension of the Euclidean space are not restrictions for our proof of the non-percolation . Besides, all results are proved for a non-random Boolean radius ℓ.

The technics for the proofs of the existence and of the non-existence of an infinite cluster drastically differ. For the existence of an infinite cluster we use technics close to the contour methods (see [1]). The non-existence is proved by a coupling of the Gibbs state and a branching process. The extinction of the branching process leads to the non-existence of infinite clusters. We use a branching process with interactions between offsprings in the same

generations and between the generations. The hard core condition prevents the accumulations of a large offspring amount which can appear because of attractive interactions between the offsprings.

The section 2 contains the definitions, all assumptions and the formulations of the main results. All proofs are in the section 3.

2 Model and Results

The configuration space, the potential function and Hamiltonian

The non-ideal gas model is a pair (Ω, φ). Here $\Omega = \{\omega\}$ is the set of all countable subsets in $\mathbb{R}^\nu$ such that for any bounded $V \subset \mathbb{R}^\nu$

$$\#(\omega \cap V) < \infty, \tag{2.1}$$

where $\#(W)$ is the number of points in W. ω is the set of points from $\mathbb{R}^\nu$, where particles $x \in \omega$ sit. The set Ω is called the set of *configurations*. We use the standard notations for the restrictions on subsets. If V is a Borel set in $\mathbb{R}^\nu$ and $\omega \in \Omega$ then $\omega_V = \omega \cap V$ and Ω_V is the set of all configurations in V. If $V \cap V' = \varnothing$ and $\omega \in \Omega$ then $\omega_{V \cup V'} =: \omega_V \vee \omega_{V'}$.

The σ-algebra $\mathfrak{A}$ in Ω is generated by the cylinder sets

$$\mathcal{A}_{V,n} := \{\omega : \#(\omega_V) = n\} \subseteq \Omega, \tag{2.2}$$

where V is a bounded Borel set in $\mathbb{R}^\nu$.

The potential function φ describes the interaction of the particles. We consider pair interactions only and assume that $\varphi(x, y)$ is continuous and satisfies the following properties.

- Translation invariance: for any $(x, y) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$ and any $z \in \mathbb{R}^\nu$ it holds that $\varphi(x + z, y + z) = \varphi(x, y)$.

  Therefore we can introduce the function $\widehat{\varphi}(x)$, $x\in \mathbb{R}^\nu$, by the equality $\widehat{\varphi}(x - y) = \varphi(x, y)$, which further we denote with the same symbol $\varphi(x)$.

- Isotropy: if B is an orthogonal operator in $\mathbb{R}^\nu$ then $\varphi(Bx) = \varphi(x)$.

- There are two reals $f \geq 0$, $d > 0$ such that $f \leq d$ and

$$\varphi(x) \begin{cases} = \infty, & \text{if } |x| \leq f, \\ \geq 0, & \text{if } |x| \in [f, d], \\ \leq 0, & \text{if } |x| \in [d, \infty). \end{cases} \tag{2.3}$$

  Besides there exists a positive monotone decreasing function ψ and $g > d$ such that

$$\varphi(x) \geq -\psi(x) \text{ for } x \geq g, \tag{2.4}$$

  and

$$I = \int_g^\infty r^{\nu-1}\psi(r)\mathrm{d}r < \infty. \tag{2.5}$$

  The condition (2.5) was proposed in [12].

- Lower boundedness: there exists $M > 0$ and x_0 such that $\min_x \varphi(x) = \varphi(x_0) = -M$.

Hamiltonian is

$$H(\omega) = \sum_{x,y\in\omega} \varphi(x-y) \tag{2.6}$$

which describes energy of configuration ω. The above expression is formal since the sum does not exists. The energy of $\omega_V \in \Omega_V$ with boundary condition $\tau \in \Omega_{V^c}$ is

$$H(\omega_V \mid \tau) = H(\omega_V) + F(\omega_V, \tau) := \sum_{x,y\in\omega_V} \varphi(x-y) + \sum_{x\in\omega_V, y\in\tau} \varphi(x-y). \tag{2.7}$$

The last sum in the above expression might be infinite. However, if $f > 0$ and (2.5) holds then $\sum_{x\in\omega_V, y\in\tau} \varphi(x-y) < \infty$ for any finite V, and any ω_V and τ.

The reference and the Gibbs measures

The reference measure Π is defined as Poisson one on $(\Omega, \mathfrak{A})$ with intensity $\lambda > 0$:

$$\Pi(\mathcal{A}_{V,n}) = \frac{\lambda^n |V|^n}{n!} e^{-\lambda |V|}, \tag{2.8}$$

where $|V|$ is the volume of V (see (2.2)). The Gibbs measure $P^{\beta,\lambda}$ on $(\Omega, \mathfrak{A})$ is determined by the Gibbs reconstruction method of the reference measure (see [4]).

To define $P^{\beta,\lambda}$ we introduce a Gibbs specification

$$\{P^{\beta,\lambda}_{V,\tau},\ V \subset \mathbb{R}^\nu, \tau \in \Omega_{V^c}\}$$

which is a family of the Gibbs reconstruction of the measure Π in finite volumes V given a conditional configuration τ and the inverse temperature $\beta \in \mathbb{R}_+$. The measure $P^{\beta,\lambda}_{V,\tau}$ has the following density $p^{\beta,\lambda}_{V,\tau}$ with respect to the measure Π :

$$p^{\beta,\lambda}_{V,\tau}(\omega_V) = \frac{\exp\{-\beta H(\omega_V \mid \tau)\}}{\int_{\Omega_V} \exp\{-\beta H(\omega \mid \tau)\} \Pi(\mathrm{d}\omega_V)}. \tag{2.9}$$

It is assumed that the densities $p^{\beta,\lambda}_{V,\tau}(\omega_V)$ are defined for boundary configurations τ such that (2.7) is finite.

We assume some conditions for the existence of the integral in (2.9) and for the existence of at least one of Gibbs measures $P^{\beta,\lambda}$ corresponding to the specification (2.9) (see [7] or [6]). If $H(\cdot|\tau)$ is finite not for all τ then the existence conditions is such that the Gibbs measure $P^{\beta,\lambda}$ is concentrated on a set of the configurations, where $H(\cdot|\tau)$ is finite, when τ from this set (see [9]).

Further we use the notation $H_V(\cdot|\cdot)$ for the energy of configurations from Ω_V with a boundary condition. We shall often omit some indices and shall write P_V instead of $P^{\beta,\lambda}_{V,\tau}$ and P instead of $P^{\beta,\lambda}$.

The percolation

Any ordered sequence of particles from a gas configuration ω we shall call a path. We say that two points $x, y \in \mathbb{R}^\nu$ ℓ-percolate with respect to ω if there exists some finite path $\pi = \{x_1, x_2, \dots, x_n\} \subset \omega$ such that $|x_i - x_{i-1}| \le \ell$ for all $i = 2, \dots, n$, and $|x - x_1| \le \ell,\ |y - x_n| \le \ell$. The set π is called a ℓ-*cluster* in ω or simply a *cluster* when ℓ and ω are fixed. We say that $x, y \in \mathbb{R}^\nu$ in a cluster π if they ℓ-percolate with respect to ω. We shall denote by $(x \rightsquigarrow^\ell y)$

the event that x and y are in a cluster. If there exists an infinite cluster starting at $x \in \mathbb{R}^\nu$, we denote this event by $(x \stackrel{\ell}{\rightsquigarrow} \infty)$. The probability of the event $(0 \stackrel{\ell}{\rightsquigarrow} \infty)$ we call *ℓ-percolation function* or *percolation function* and denote $\theta_\ell(\beta, \lambda)$:

$$\theta_\ell(\beta, \lambda) := P^{\beta,\lambda}(0 \stackrel{\ell}{\rightsquigarrow} \infty). \tag{2.10}$$

The main result on the non-percolation

The result on the non-percolation is proved for $f > 0$ (the hard-core condition) and any ν (any dimension of the Euclidean space).

Theorem 2.1. *For any $\ell > f$ there exists λ_ℓ^-, $0 < \lambda_\ell^- < \infty$, and a function $\beta_\ell^-(\lambda)$ defined on the interval $(0, \lambda_\ell^-)$ such that*

1. *$0 < \beta_\ell^-(\lambda) < \infty$ on the interval $(0, \lambda_\ell^-)$,*
2. *$\beta_\ell^-(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$.*

Let

$$A_- = \{(\lambda, \beta) : \ \lambda < \lambda_\ell^-, \ \beta < \beta_\ell^-(\lambda)\}.$$

Then all clusters are finite with the probability 1 if $(\lambda, \beta) \in A_-$, that is $\theta_\ell(\beta, \lambda) = 0$ for those (λ, β) (see Fig.1). Moreover the expectation of the cluster size is finite.

The main result on the percolation

The next theorem is proved for the case $\nu = 2$. Now the Boolean radius is bounded below, however the hard core is not necessary.

Theorem 2.2. *For any $\ell > 2\sqrt{2}d$ there exist λ_ℓ^+, $0 < \lambda_\ell^+ < \infty$, and a function $\beta_\ell^+(\lambda)$ defined on the interval $(0, \lambda_\ell^+)$ such that*

1. *$0 < \beta_\ell^+(\lambda) < \infty$ on the interval $(0, \lambda_\ell^+)$,*
2. *$\beta_\ell^+(\lambda) \uparrow \infty$ as $\lambda \downarrow 0$.*

Let

$$A_+ = \{(\lambda, \beta) : \ \lambda < \lambda_\ell^+, \beta > \beta_\ell^+(\lambda)\} \cup \{(\lambda, \beta) : \ \lambda > \lambda_\ell^+, \beta \geq 0\}.$$

Then with the probability 1 there exists an infinite cluster if $(\lambda, \beta) \in A_+$, that is $\theta_\ell(\beta, \lambda) > 0$ for those (λ, β) (see Fig.1).

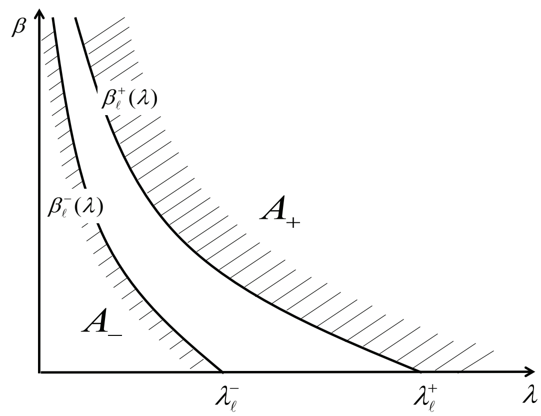

Figure 1: Percolation and non-percolation regions.

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The next corollary joins the results of theorems 2.1 and 2.2. However, the claims of the corollary hold under the most restrictive assumptions from the assumptions of both theorems.

Corollary 2.3. *Let $\nu = 2$ and $f > 0$. For all $\ell > 2\sqrt{2}d$ there exist two positive reals λ_ℓ^- and λ_ℓ^+, and two functions: $\beta_\ell^-(\lambda)$ defined on $(0, \lambda_\ell^-)$, and $\beta_\ell^+(\lambda)$ defined on $(0, \lambda_\ell^+)$, such that*

1. $0 < \lambda_\ell^- < \lambda_\ell^+ < \infty$,
2. $0 < \beta_\ell^-(\lambda) < \infty$ *and* $0 < \beta_\ell^+(\lambda) < \infty$ *on their intervals of the definitions,*
3. $\beta_\ell^-(\lambda) < \beta_\ell^+(\lambda)$ *on* $(0, \lambda_\ell^-)$,
4. $\beta_\ell^-(\lambda) \uparrow \infty$ *and* $\beta_\ell^+(\lambda) \uparrow \infty$ *as* $\lambda \downarrow 0$.

Let

$$A_- = \{(\lambda, \beta) : \lambda < \lambda_\ell^-,\ \beta < \beta_\ell^-(\lambda)\}$$

and

$$A_+ = \{(\lambda, \beta) : \lambda < \lambda_\ell^+,\ \beta > \beta_\ell^+(\lambda)\} \cup \{(\lambda, \beta) : \lambda > \lambda_\ell^+,\ \beta \geq 0\}.$$

Then

1. *with the probability 1 all clusters are finite if* $(\lambda, \beta) \in A_-$*,*

2. *with the probability 1 there exists an infinite cluster if* $(\lambda, \beta) \in A_+$*,*

(see Fig.1).

Moreover the expectation of the cluster size is finite if $(\lambda, \beta) \in A_-$*.*

3 Proofs

3.1 The proof of the non-percolation

It is essential for the proof that the potential function has the hard core. However, the arguments in this subsection do not depend on the dimension of the space.

The next lemma is an obvious consequence of the hard core condition and the inequality (2.5).

Lemma 3.1. *Let* $V \subseteq \mathbb{R}^\nu$ *be a finite volume and let* σ *be a configuration in* V*. Then*

$$\int_{V^c} e^{-H(\sigma\vee\omega)}\Pi(\mathrm{d}\omega) < \infty. \tag{3.1}$$

We obtain the non-percolation result by a coupling of the Gibbs measure $P^{\beta,\lambda}$ with a *branching cluster process.* The method was described in the work [8]. Here we use the analogous idea.

Note that due to the hard core condition, one considers the ℓ-percolation only when $\ell > f$. Informally the idea of the proof is the following. Suppose that $R \subset \omega$ is a ℓ-cluster with $\#(R) > 1$. Let us choose some particle x_0 from this cluster, $x_0 \in R$. Let $R^{(1)} \subset R$ be the set of all particles from R such that the distance between x_0 and any point from $R^{(1)}$ is less than or equal to ℓ. The set $R^{(1)}$ is not empty because $\#(R) > 1$. Next we choose a set $R^{(2)}$, where $R^{(2)} \subset R \setminus (R^{(1)} \cup \{x_0\})$, and which is the set of the particles such that for any v from $R^{(2)}$ there exists at least one point w from $R^{(1)}$ at the distance no greater than ℓ. We can call v an *offspring* of w. If the set $R \setminus (R^{(2)} \cup R^{(1)} \cup \{x_0\})$ is not empty we can choose a subset $R^{(3)}$ with the similar properties, ect. Iterating the procedure we will obtain the following representation of the cluster $R: \ R = \cup_{i=0}^{\infty} R^{(i)}$ (here $R^{(0)} = \{x_0\}$). The set $R^{(i)}$ we call *i-th generation.* The set $R^{(n-1)}$ generates the set $R^{(n)}$.

This branching construction brings us the idea of a branching process, but there are two peculiarities that differ our process from the ordinary branching

process. First, note that it is possible for one offspring to have different parents. Thus we do not have here a branching tree and it means that the independence of the offsprings does not hold. Second, keeping in mind the coupling, we define the transition probabilities of the branching process by Gibbs measure P_V. Thus for ℓ sufficiently small it is possible that some generation $R^{(n)}$ interacts with precedent generations $R^{(k)}, k < n$.

Next we describe a rigorous construction of the cluster branching process.

3.1.1 The cluster branching process

We describe a path of the process and its distribution P.

Let x_0 be some point from $\mathbb{R}^\nu$. We shall construct a sequence $(R^{(n)})$ which describes the generations. Together with the sequence $(R^{(n)})$ we define the sequence (E_n), where $E_n = \cup_{i=0}^n R^{(i)}$ and the sequence of the *occupied areas* $B_n = \cup_{v \in E_{n-1}} B_\ell(v)$. The set E_n we call an *environment*.

Initial step. Let $R^{(0)} = \{x_0\}$ and $E_0 = \{x_0\}$, $B_0 = \emptyset$.

First step. Let $R^{(1)} = \{x_1^{(1)}, \ldots, x_{k_1}^{(1)}\}$ be a set of the particles in the ball $B_\ell(x_0)$ with the center x_0 and with the radius ℓ. The set $R^{(1)}$ is the offspring set of x_0. Then $E_1 = E_0 \cup R^{(1)}$ and $B_1 = B_\ell(x_0)$. B_1 is the occupied area by the offsprings of x_0. No particles of further embranchments appear in B_1. We define a conditional probability density ρ of the measure P with respect to the same Poisson measure Π. The density with respect to Π of the offspring set $R^{(1)}$ of the ancestor x_0 is

$$\rho(x_1^{(1)}, \ldots, x_{k_1}^{(1)} \mid E_0) = \frac{1}{Z(E_0)} \int_{\Omega_{B_1^c}} e^{-\beta H(E_1 \vee \omega)} \Pi(\mathrm{d}\omega), \tag{3.2}$$

where $\Omega_{B_1^c}$ is the set of all configurations where particles "live" outside of the ball B_1, and

$$Z(E_0) = \int_\Omega e^{-\beta H(E_0 \vee \omega)} \Pi(\mathrm{d}\omega).$$

We use Gibbs measure $P^{\beta,\lambda}$ for the definition of ρ. In fact, all defined probabilities and further calculations assume a big volume V, where we consider all configurations. It means that we use the measure P_V instead $P^{\beta,\lambda}$. For example, in (3.2) the integration is taken over $\Omega_{B_1^c \cap V}$. Therefore ρ depends on V. However, in what follows all estimates do not depend on V, and hence can be considered as the estimates in the infinite volume. We do not mention the volume V in the further calculations except cases when it is required.

Using ρ we can calculate, for example, the probability to have k offsprings of x_0 :

$$\begin{aligned}\mathsf{P}(\#(R^{(1)}) = k \mid E_0) &= \int_{\{R^{(1)}\in\Omega_{B_1}:\ \#(R^{(1)})=k\}} \rho(R^{(1)} \mid E_0)\Pi(\mathrm{d}R^{(1)}) \\ &= \frac{\lambda^k |B_1|^k}{k!} e^{-\lambda|B_1|} \int_{(B_1)^k} \rho(x_1^{(1)}, \dots, x_k^{(1)} \mid E_0)\mathrm{d}x_1^{(1)} \dots \mathrm{d}x_k^{(1)}.\end{aligned}$$

Second step. In this step we describe the embranchments of all particles from $R^{(1)}$. Any particle branches according to some order introduced in $R^{(1)}$. We shall construct the set $R^{(2)}$ in according to the chosen order. Let $R^{(2,1)} = \{x_1^{(2,1)}, \dots, x_{k_{2,1}}^{(2,1)}\} \subset B_\ell(x_1^{(1)}) \setminus B_1$ be the set of offsprings of $x_1^{(1)}$. The offsprings of $x_1^{(1)}$ cannot be situated in B_1 since B_1 is occupied by the offsprings of x_0. Let $E_{(2,1)} = E_1 \cup R^{(2,1)}$ and $B_{(2,1)} = B_1 \cup B_\ell(x_1^{(1)})$. The probability density of the offsprings of $x_1^{(1)}$ is

$$\rho(x_1^{(2,1)}, \dots, x_{k_{2,1}}^{(2,1)} \mid E_1) = \frac{1}{Z(E_1)} \int_{\Omega_{(B_{(2,1)})^c}} e^{-\beta H(R^{(2,1)} \vee E_1 \vee \omega)} \Pi(\mathrm{d}\omega), \tag{3.3}$$

where

$$Z(E_1) = \int_{\Omega_{B_1^c}} e^{-\beta H(E_1 \vee \omega)} \Pi(\mathrm{d}\omega) \tag{3.4}$$

(see (3.1)).

Suppose now that $k < k_1$ particles from $R^{(1)}$ are already branched and $R^{(2,1)}, \dots, R^{(2,k)}$ is a sequence of their offsprings. Hence we have the environment (all already living particles) $E_{(2,k)} = E_1 \cup \bigcup_{i=1}^k R^{(2,i)}$ and the occupied area $B_{(2,k)} = B_1 \cup \bigcup_{i=1}^k B_\ell(x_i^{(1)})$. Note that $B_{(2,k)}$ is the ℓ-neighborhood of $E_0 \cup \{x_1^{(1)}, \dots, x_k^{(1)}\}$.

Let now the next point $x_{k+1}^{(1)}$ be branching. Let

$$R^{(2,k+1)} = \left\{ x_1^{(2,k+1)}, \dots, x_{k_{2,k+1}}^{(2,k+1)} \right\} \subset B_\ell(x_{k+1}^{(1)}) \setminus B_{(2,k)}$$

be the set of offsprings of $x_{k+1}^{(1)}$. Then $E_{(2,k+1)} = E_{(2,k)} \cup R^{(2,k+1)}$ and $B_{(2,k+1)} = B_{(2,k)} \cup B_\ell(x_{k+1}^{(1)})$. The probability density of the offsprings is

$$\rho\left(x_1^{(2,k+1)}, \dots, x_{k_{2,k+1}}^{(2,k+1)} \mid E_{(2,k)}\right) = \frac{1}{Z(E_{(2,k)})} \int_{\Omega_{(B_{(2,k+1)})^c}} e^{-\beta H(E_{(2,k+1)} \vee \omega)} \Pi(\mathrm{d}\omega). \tag{3.5}$$

We obtain the next generation $R^{(2)} = \bigcup_{i=1}^{k_1} R^{(2,i)}$ after the embranchments of all points from $R^{(1)}$. Let $E_2 = E_{(2,k_1)}$ and $B_2 = B_{(2,k_1)}$.

$(n+1)$*-th step.* To construct $R^{(n+1)}$ from $R^{(n)}$ we follow the same scheme. Let $R^{(n)} = \{x_1^{(n)}, \dots, x_{k_n}^{(n)}\}$. The particles from $R^{(n)}$ are branching according to some chosen order in $R^{(n)}$. Suppose that k ($k < k_n$) particles from $R^{(n)}$ are branched. Hence we have the sets $R^{(n+1,i)}$ (where $i \le k$), $E_{(n+1,k)}$ and $B_{(n+1,k)}$.

Now, let

$$R^{(n+1,k+1)} = \left\{x_1^{(n+1,k+1)}, \dots, x_{k_{n+1,k+1}}^{(n+1,k+1)}\right\} \subset B_\ell(x_{k+1}^{(n)}) \setminus B_{(n+1,k)}$$

be the set of offsprings of the branching particle $x_{k+1}^{(n)}$. Then $E_{(n+1,k+1)} = E_{(n+1,k)} \cup R^{(n+1,k+1)}$ and $B_{(n+1,k+1)} = B_{(n+1,k)} \cup B_\ell(x_{k+1}^{(n)})$. The probabilistic density is

$$\begin{aligned} \rho &\left(x_1^{(n+1,k+1)}, \dots, x_{k_{n+1,k+1}}^{(n+1,k+1)} \mid E_{(n+1,k)}\right) \\ &= \frac{1}{Z(E_{(n+1,k)})} \int_{\Omega_{(B_{(n+1,k+1)})^c}} e^{-\beta H(E_{(n+1,k+1)} \vee \omega)} \Pi(\mathrm{d}\omega). \end{aligned} \quad (3.6)$$

The above iterative steps describe a path and the transition probabilities of the cluster growth process. Note that the offsprings can depend not only on the preceding generation, but on *all* previous generations.

The question we are interested is when the cluster branching process extincts with the probability 1. It is known that the extinction condition of the ordinary branching processes is formulated in the term of the mean number of the offsprings. We can expect that the extinction of the cluster branching process is controlled by the offspring mean value, as well. We show further that if the mean number of the offsprings is uniformly less than 1 over all possible cluster configurations of the previous generations, then the cluster branching process will extinct with the probability 1.

In the next lemma we prove that there exists a region in the plane (λ, β) such that the mean offspring number of one ancestor is less than 1. Then, we show that this condition is sufficient for the extinction of the cluster branching process. Moreover we show that the mean value of paths of the cluster branching process is finite.

Lemma 3.2. *There exists λ_ℓ^- and a function $\beta^-(\lambda)$ such that for any $\lambda < \lambda_\ell^-$ and $\beta < \beta^-(\lambda)$ the expected number of the offsprings $\#(R^{(n,k)})$ of the ancestor $x_k^{(n-1)}$ is less than 1, uniformly over n, k and over the environment $E_{(n,k-1)}$.*

$$\sup_{n,k} \mathsf{E}\big(\#(R^{(n,k)})\ \big|E_{(n,k-1)}\big) < 1. \tag{3.7}$$

Proof We give an estimate of the probability the point $x_k^{(n-1)}$ to have exactly K offsprings, that is $\#(R^{n,k}) = K$. By the definition of the offspring density (3.6) we have to estimate the following integral. Let $\widetilde{B}(x_k^{(n-1)}) := B_\ell(x_k^{(n-1)}) \setminus B_{(n,k-1)}$ then

$$\begin{aligned} &\mathsf{P}\big(\#(R^{(n,k)}) = K\ \big|\ E_{(n,k-1)}\big) \\ &= \int_{\left\{R^{(n,k)}\in\Omega_{\widetilde{B}\left(x_k^{(n-1)}\right)}:\#(R^{(n,k)})=K\right\}} \rho(R^{(n,k)}\mid E_{(n,k-1)})\Pi(\mathrm{d}R^{(n,k)}) \end{aligned} \tag{3.8}$$

We shorten some notations in the further calculations. Let $\widetilde{R} := R^{(n,k)}$ and $\widetilde{E} := E_n^{(k-1)}$. Than the integral (3.8) can be represented as

$$\begin{aligned} &\mathsf{P}(\#(R^{(n,k)}) = K \mid E_{(n,k-1)}) = \mathsf{P}(\#(\widetilde{R}) = K \mid \widetilde{E}) \\ &= \frac{1}{Z(\widetilde{E})} \int_{\left\{\widetilde{R}\in\Omega_{\widetilde{B}\left(x_k^{(n-1)}\right)}:\#(\widetilde{R})=K\right\}} \int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{R}\vee\widetilde{E}\vee\omega)}\Pi(\mathrm{d}\omega)\Pi(\mathrm{d}\widetilde{R}), \end{aligned} \tag{3.9}$$

where

$$Z(\widetilde{E}) = \int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{E}\vee\omega)}\Pi(\mathrm{d}\omega). \tag{3.10}$$

Hamiltonian in (3.9) can be represented as

$$H(\widetilde{R}\vee\widetilde{E}\vee\omega) = H(\widetilde{R}) + H(\widetilde{E}\vee\omega) + F(\widetilde{R}, \widetilde{E}\vee\omega). \tag{3.11}$$

Let C_m be a sequence of the strips

$$\mathsf{C}_m = \{x \in \mathbb{R}^\nu :\ m \le |x - x_k^{(n-1)}| < m+1\},$$

where $m \in \mathbb{Z}_+$ and $m \ge g$, and

$$\mathsf{C}_0 = \{x \in \mathbb{R}^\nu :\ \ell \le |x - x_k^{(n-1)}| < m_0\},$$

where $m_0 = \min\{m: m \geq g\}$. The volume of C_m is

$$|\mathsf{C}_m| = \kappa P_{\nu-1}(m),$$

where $P_{\nu-1}(r)$ is a polynomial having its power $\nu - 1$, and the leading coefficient is $\nu - 1$. Let $\Omega_m = \Omega_{\mathsf{C}_m \setminus B_{(n,k)}}$. Let $\omega \in \Omega_m$ then the number of the particles of the configuration $\omega \vee \widetilde{E}$ in C_m has the following bound

$$\#\left((\omega \vee \widetilde{E})_{\mathsf{C}_m}\right) \leq \left[\frac{|\mathsf{C}_m|}{\kappa\left(\frac{f}{2}\right)^\nu}\right] \leq \frac{P_{\nu-1}(m)}{\left(\frac{f}{2}\right)^\nu},$$

where $(\omega \vee \widetilde{E})_{\mathsf{C}_m}$ is the restriction of $\omega \vee \widetilde{E}$ to C_m. We have then the following lower bound for the interaction energy of $(\omega \vee \widetilde{E})_{\mathsf{C}_m}$ and $\widetilde{R}$ consisting K offsprings, $\#(\widetilde{R}) = K$,

$$F\left((\omega \vee \widetilde{E})_{\mathsf{C}_m}, \widetilde{R}\right) \geq -\psi(m) K \frac{2^\nu P_{\nu-1}(m)}{f^\nu}.$$

The energy of the interaction of $\omega \vee \widetilde{E}$ and $\widetilde{R}$ is estimated as

$$\begin{aligned} F\left(\omega \vee \widetilde{E}, \widetilde{R}\right) \geq & \quad - KMn_0 - \frac{2^\nu K}{f^\nu} \sum_{m=m_0}^{\infty} P_{\nu-1}(m)\psi(m) \\ \geq & \quad - KMn_0 - \frac{2^\nu K}{f^\nu} \int_{m_0}^{\infty} P_{\nu-1}(r)\psi(r)\mathrm{d}r \\ \geq & \quad - K\left(Mn_0 + \frac{2^\nu}{f^\nu} I\right), \end{aligned}$$

where $n_0 = \left[\frac{|\mathsf{C}_0|}{\kappa\left(\frac{f}{2}\right)^\nu}\right]$ (see (2.5)). Let $n_1 = Mn_0 + \frac{2^\nu}{f^\nu} I$.

By (3.9) and (3.11) we have

$$\begin{aligned} & \mathsf{P}(\widetilde{R} : \#(\widetilde{R}) = K \mid \widetilde{E}) \\ & \leq \frac{e^{\beta K n_1}}{Z(\widetilde{E})} \int_{\left\{\widetilde{R} \in \Omega_{\widetilde{B}\left(x_k^{(n-1)}\right)} : \#(\widetilde{R}) = K\right\}} e^{-\beta H(\widetilde{R})} \Pi(\mathrm{d}\widetilde{R}) \int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{E} \vee \omega)} \Pi(\mathrm{d}\omega). \end{aligned}$$

The cluster energy $H(\widetilde{R})$ we can estimate very roughly as

$$- H(\widetilde{R}) < M\binom{K}{2} < MK^2. \tag{3.12}$$

Thus

$$
\begin{aligned}
&\mathsf{P}(\widetilde{R}:\#(\widetilde{R})=K \mid \widetilde{E}) \\
&\le \frac{e^{\beta(Kn_1+MK^2)}}{Z(\widetilde{E})}\int_{\left\{\widetilde{R}\in\Omega_{\widetilde{B}\left(x_k^{(n-1)}\right)}:\#(\widetilde{R})=K\right\}}\Pi(\mathrm{d}\widetilde{R})\int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{E}\vee\omega)}\Pi(\mathrm{d}\omega) \\
&\le \frac{e^{\beta(Kn_1+MK^2)}}{Z(\widetilde{E})}\frac{(\lambda|\widetilde{B}(x_k^{(n-1)})|)^K}{K!}e^{-\lambda|\widetilde{B}(x_k^{(n-1)})|}\int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{E}\vee\omega)}\Pi(\mathrm{d}\omega) \\
&\le e^{\beta(Kn_1+MK^2)}\frac{(\lambda|B_\ell(x_k^{(n-1)})|)^K}{K!}. \qquad (3.13)
\end{aligned}
$$

In the last inequality we use

$$\frac{1}{Z(\widetilde{E})}\int_{\Omega_{B^c_{(n,k)}}} e^{-\beta H(\widetilde{E}\vee\omega)}\Pi(\mathrm{d}\omega)=1.$$

Besides in the last inequality in (3.13) we estimate the volume $|\widetilde{B}(x_k^{(n-1)})|$ by $|B_\ell(x_k^{(n-1)})|$. Note that $\widetilde{B}(x_k^{(n-1)})$ can be empty, that means that the particle $x_k^{(n-1)}$ has no offsprings. It leads to zero of the probability we are estimating. We do not use this possibility.

The right hand side of (3.13) does not depend on the volume V.

The maximal number of the particles in the ball $B_\ell(x_k^{(n-1)})$ is $n_B = |B_\ell(x_k^{(n-1)})|/(\kappa(f/2)^\nu)$. Thus using estimation (3.13) we can estimate the mean number of the offsprings:

$$
\begin{aligned}
\mathsf{E}\big(\#(\widetilde{R})\ \big|\ \widetilde{E}\big) &\le \sum_{k=0}^{n_B} k e^{\beta k n_1+\beta k^2 M}\frac{(\lambda|B_\ell(x_k^{(n-1)})|)^k}{k!} \\
&\le \lambda\kappa\ell^\nu e^{\beta n_B n_1+\beta n_B^2 M}e^{\lambda\kappa\ell^\nu}. \qquad (3.14)
\end{aligned}
$$

Let

$$\beta_\ell^-(\lambda) = -\frac{1}{A}\ln\lambda-\frac{\kappa\ell^\nu}{A}\lambda-\ln\left(\frac{\kappa\ell^\nu}{A}\right), \qquad (3.15)$$

where

$$A=n_B(Mn_B+n_1).$$

The function $\beta^-(\lambda)$ is considered on the interval $(0,\lambda_\ell^-]$, where λ_ℓ^- is a root of the equation

$$-\ln\lambda-\kappa\ell^\nu\lambda-\ln(\kappa\ell^\nu)=0. \qquad (3.16)$$

When $\lambda < \lambda_\ell^-$ and $\beta < \beta^-(\lambda)$ we obtain

$$\mathsf{E}\big(\#(\widetilde{R}) \bigm| \widetilde{E}\big) < 1 \tag{3.17}$$

uniformly over the environment $\widetilde{E}$. $\square$

Next we show that if the mean number of the offsprings is less than 1 (see (3.17)) then the mean size of the cluster is finite. Indeed, if $\beta < \beta^-(\lambda)$, then there exists $\epsilon > 0$ depending on β and λ such that $\mathsf{E}(\#(\widetilde{R}) \mid \widetilde{E}) < 1 - \epsilon$, and

$$\begin{aligned}
\mathsf{E}(\#(R^{(n)})) &= \mathsf{E}\Big(\sum_{k=1}^{\infty} I_{\{\#(R^{(n-1)})=k\}} \#(R^{(n)})\Big) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathsf{E}\Big(I_{\{\#(R^{(n-1)})=k\}} \#(R^{(n,i)})\Big) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathsf{E}\Big(I_{\{\#(R^{(n-1)})=k\}} \mathsf{E}\big(\#(R^{(n,i)}) \big| E_{(n,i-1)}\big)\Big) \\
&\le (1-\epsilon)\sum_{k=1}^{\infty}\sum_{i=1}^{k} \mathsf{E}(I_{\{\#(R^{(n-1)})=k\}}) = (1-\epsilon)\mathsf{E}(\#(R^{(n-1)})).
\end{aligned}$$

It means that

$$\mathsf{E}(\#(R^{(n)})) < (1-\epsilon)^n,$$

and we see that the mean cluster size is finite,

$$\mathsf{E}\Big(\sum_{n=1}^{\infty} \#(R^{(n)})\Big) = \sum_{n=1}^{\infty} E(\#(R^{(n)})) \le 1/\epsilon. \tag{3.18}$$

Since the path $R^{(n)}$ is a ℓ-connected set of points in V and the estimate (3.18) does not depend on V, the probability of infinite clusters is 0.

This proves that the cluster branching process is degenerated.

Any path of the cluster branching process starting from x_0 is

$$E = \cup_{n=1}^{N} R^{(n)},$$

where N is the number of the generations. N is finite since (3.18). The relation (3.18) can be rewritten as

$$\mathsf{E}(\#(E)) = \sum_k k \int_{\{E:\ \#(E)=k\}} \rho(E \mid E_0)\Pi(\mathrm{d}E) < \infty. \tag{3.19}$$

The coupling

We explain next the coupling of the Gibbs field having the distribution $P^{\beta,\lambda}$ and the branching cluster process having the distribution P. To this end we represent any finite ℓ-cluster of a configuration ω as a path of the branching cluster process.

Let $\gamma \subset \omega_0 \in \Omega$ be a finite ℓ-cluster in a configuration ω_0. It means that $B_{\ell/2}(\gamma) = \cup_{x\in\gamma} B_{\ell/2}(x)$ is connected component in $B_{\ell/2}(\omega_0) = \cup_{x\in\omega_0} B_{\ell/2}(x)$.

Choose a particle $x_0 \in \gamma$. We can consider the configuration γ as a branching process path $E = (E_n)$, $E_n \subseteq E_{n+1}$, starting at the particle x_0 ($E_0 = \{x_0\}$), and such that $\gamma = \cup E_n$. The construction was described in the section 3.1.1. By the described iteration we obtain also a sequence of generations $(R^{(n)})$ such that $\gamma = \cup_n R^{(n)}$. Any generation $R^{(n)}$ is a set of the offsprings of $R^{(n-1)}$.

Next we find a relation between the distribution of the cluster branching process and the Gibbs measure. Consider the event

$$\Delta_\gamma = \{\omega : \ \gamma \subset \omega, \ B_\ell(\gamma) \cap \gamma = \gamma\}.$$

We can define a density χ of this event with the respect to Π.

$$\chi(\Delta_\gamma) = \frac{1}{Z_V} \int_{\Omega_{(B_\ell(\gamma))^c}} e^{-\beta H(\gamma\vee\omega)} \Pi(\mathrm{d}\omega) \tag{3.20}$$

The density $\rho(\gamma)$ of the path γ is

$$\rho(\gamma \mid E_0) = \prod_{n=1}^{N} \rho(R^{(n)} \mid E_{n-1}). \tag{3.21}$$

It follows from finiteness of γ that N is finite.

It is not difficult to verify that

$$\rho(\gamma \mid E_0) = \frac{Z(E_N)}{Z(E_0)}. \tag{3.22}$$

Then

$$\chi(\Delta_\gamma) = \rho(\gamma \mid E_0) \frac{Z(E_0)}{Z_V}. \tag{3.23}$$

Let (V_m) and $(\widetilde{V}_m)$ be sequences of the boxes

$$V_m = \{x \in \mathbb{R}^\nu : \ |x| \le m\}, \ \ \widetilde{V}_m = \{x \in \mathbb{R}^\nu : \ |x| \le m - \ell\}.$$

Define the sequence (γ_m) where $\gamma_m = \gamma \cap \widetilde{V}_m$. For any γ_m we have the relation (3.23).

Finiteness of the expectation of the ℓ-clusters follows from (3.23). Let

$$\Gamma_{x_0}^k = \{\gamma \subset \mathbb{R}^\nu : \ \#(\gamma) = k, \ x_0 \in \gamma\}$$

be the set of all ℓ-clusters containing the particle x_0 and having exactly k particles.

Consider the set of the configurations

$$\Omega^{\Gamma_{x_0}^k} = \bigcup_{\gamma \in \Gamma_{x_0}^k} \Delta_\gamma$$

and its Gibbs probability

$$\begin{aligned} P_V\big(\Omega^{\Gamma_{x_0}^k}\big) &= \frac{1}{Z_V} \int_{\Omega^{\Gamma_{x_0}^k}} e^{-\beta H(\omega)} \Pi(\mathrm{d}\omega) \\ &= \frac{1}{Z_V} \int_{\Gamma_{x_0}^k} \int_{\Omega_{(B_\ell(\gamma))^c}} e^{-\beta H(\omega \vee \gamma)} \Pi(\mathrm{d}\omega) \Pi(\mathrm{d}(\gamma \vee \phi_{B_\ell(\gamma)})), \end{aligned}$$

where $\phi_{B_\ell(\gamma)}$ is the empty configuration in the ℓ-neighborhood of γ. Hence

$$P_V\big(\Omega^{\Gamma_{x_0}^k}\big) = \int_{\Gamma_{x_0}^k} \rho(\gamma \mid x_0) \Pi(\mathrm{d}(\gamma \vee \phi_{B_\ell(\gamma)}))$$

The mean value of the size of the clusters γ then is

$$\begin{aligned} E_V(\#(\gamma)) &= \sum_{k=1}^{\infty} k P_V\big(\Omega^{\Gamma_{x_0}^k}\big) \\ &= \sum_{k=1}^{\infty} k \int_{\Gamma_{x_0}^k} \rho(\gamma \mid x_0) \Pi(\mathrm{d}(\gamma \vee \phi_{B_\ell(\gamma)})) < \infty \end{aligned}$$

since (3.19). This proves Theorem 2.1 $\square$

3.2 The proof of the percolation

The proof of the existence of infinite clusters is based on technics which is close to the contour method in the lattice models. To apply the method we discretize $\mathbb{R}^2$ splitting it into squares. A c-contour around 0 is a set of

empty (without particles) squares surrounding 0. The main fact we prove is that the probability of a c-contour decreases exponentially with its length. It leads to the finiteness of the number of the contours surrounding 0.

In our proof of the percolation, the essential assumption is that the space is two-dimension. The hard core is not used.

Divide $\mathbb{R}^2$ into square cells $\mathcal{S} = \{S^q_{(k,l)}\}$ of the linear size equal to q. Suppose that the left-lower corner of any cell $S^q_{(k,l)}$ has coordinate (kq, lq), where $(k,l) \in \mathbb{Z}^2$. So we have a natural order of the cells. The point $c_{(k,l)} = \left(\frac{2k+1}{2}q, \frac{2l+1}{2}q\right)$ is called the center of the cell $S^q_{(k,l)}$. Two cells $S^q_{(k,l)}$ and $S^q_{(k',l')}$ are neighbors if either $k = k' \pm 1$ and $l = l'$ or $l = l' \pm 1$ and $k = k'$. Let $\langle c, c' \rangle$ be the line connecting the centers $c = c_{(k,l)}$ and $c' = c_{(k',l')}$ if $S^r_{(k,l)}$ and $S^q_{(k',l')}$ are neighbors. Let $\mathcal{P} = \{S^q_{(k,l)}\} \subseteq \mathcal{S}$ be a finite subset of the cells and $\mathbf{C}(\mathcal{P}) = \{c_{(k,l)} : S^q_{(k',l')} \in \mathcal{P}\}$ be the set of all centers of the cells from $\mathcal{P}$. For every set $\mathcal{P}$ of the cells we consider the graph

$$G_{\mathcal{P}} = \Big(\mathbf{C}(\mathcal{P}), \Gamma(\mathcal{P}) = \{\langle c, c' \rangle : c, c' \in \mathbf{C}(\mathcal{P})\}\Big)$$

having $\mathbf{C}(\mathcal{P})$ as the vertex set and $\Gamma(\mathcal{P})$ as the bond set of all bonds connecting neighboring cells from $\mathcal{P}$. A set of cells $\mathcal{P}$ is connected if the graph $G_{\mathcal{P}}$ is connected.

A set of cells $\mathcal{R}$ is called *contour* if the bond set $\Gamma(\mathcal{R})$ is homeomorphic to the circle. The number $n(\mathcal{R})$ of the cells in a contour $\mathcal{R}$ is called the length of the contour.

If $\mathcal{P}$ is a set of cells then $W(\mathcal{P}) = \bigcup_{S \in \mathcal{P}} S \subseteq \mathbb{R}^2$ is the support of $\mathcal{P}$.

All contours we consider further surround $0 \in \mathbb{R}^2$. Therefore we often omit mentioning this. Let $\omega \in \Omega$ be a configuration. If a contour $\mathcal{R}$ is such that $\omega \cap W(\mathcal{R}) = \varnothing$ then we call it a *c-contour with the respect to ω* or simply a *c-contour*.

The proof of Theorem 2.2 is based on the following lemma. Let $\Omega^0(\mathcal{R})$ be the event (the set of configurations) such that the contour $\mathcal{R}$ is the c-contour with respect to any $\omega \in \Omega^0(\mathcal{R})$, that is $\Omega^0(\mathcal{R}) = (\omega_{W(\mathcal{R})} = \varnothing)$. Let $\widetilde{\Omega} = \Omega_{W(\mathcal{R})^c}$ be the set of all configurations out of $W(\mathcal{R})$. Then

$$\Omega^0(\mathcal{R}) = \{\omega = \phi_{W(\mathcal{R})} \vee \widetilde{\omega} : \widetilde{\omega} \in \widetilde{\Omega}\},$$

where $\phi_{W(\mathcal{R})}$ is the empty configuration in $W(\mathcal{R})$.

Lemma 3.3. *Let the cell size be $q = 2d + \delta$, where δ is a small positive number. There exist constants $\alpha \in (0, 1)$, $c(\beta) > 0$ and $G(\beta, \lambda)$ such that for any $h \geq 0$ there exists a function $\beta^{+}_{\ell,h}(\lambda) \geq 0$ defining the domain*

$$\{(\lambda, \beta) : \ \beta > \beta^{+}_{\ell,h}(\lambda)\}$$

where the following relations hold:

1. *$G(\beta, \lambda) > h$,*
2. *for any c-contour $\mathcal{R}$*

$$P^{\beta,\lambda}(\Omega^0(\mathcal{R})) < c(\beta) e^{-n(\mathcal{R})\alpha G\beta,\lambda)}. \tag{3.24}$$

The probability that there are no particles in set $W(\mathcal{R})$ which is the support of the c-contour $\mathcal{R}$ exponentially decreases with the contour length.

Proof Let V be a volume in $\mathbb{R}^2$ containing $W(\mathcal{R}) = \bigcup_{S \in \mathcal{R}} S$. In order to estimate the probability of event $\Omega^0(\mathcal{R})$ (see Figure 3-A) we construct a event $\Omega^1(\mathcal{R})$ by adding particles in the c-contour $\mathcal{R}$ (see Figure 3-B). That allows to obtain the lower bound of probability $P_V(\Omega^1(\mathcal{R}))$ of the form $P_V(\Omega^1(\mathcal{R})) > e^{n\alpha G(\beta,\lambda)} P_V(\Omega^0)$, where $n = n(\mathcal{R})$. Substituting the probability $P_V(\Omega^1(\mathcal{R}))$ by 1, we immediately obtain (3.24).

The probability of $\Omega^0(\mathcal{R})$

Recall that $\Omega^0(\mathcal{R})$ is the event composed of the configurations in V containing c-contour $\mathcal{R}$. We assume that the boundary configuration out of V is $\tau = \varnothing$. The probability of the event then is

$$\begin{aligned} P_V(\Omega^0(\mathcal{R})) &= \frac{1}{Z_V} \int_{\Omega^0(\mathcal{R})} \exp\{-\beta H(\omega)\} \Pi(\mathrm{d}\omega) \\ &= \frac{e^{-\lambda \Delta n}}{Z_V} \int_{\widetilde{\Omega}} \exp\{-\beta H(\widetilde{\omega})\} \Pi(\mathrm{d}\widetilde{\omega}), \end{aligned} \tag{3.25}$$

where $\Delta = (2d+\delta)^2$ is the volume of any cell, $\widetilde{\Omega}$ is the set of all configurations in $V \setminus W(\mathcal{R})$.

Let $\phi_{W(\mathcal{R})}$ be the empty configuration in the region $W(\mathcal{R})$. Any configuration $\omega \in \Omega^0(\mathcal{R})$ is the composition of $\phi_{W(\mathcal{R})}$ and a configuration $\widetilde{\omega}$ in $V \setminus W(\mathcal{R})$, $\omega = \phi_{W(\mathcal{R})} \vee \widetilde{\omega}$. The pre-integral factor in (3.25) is the integration result over $\phi_{W(\mathcal{R})}$.

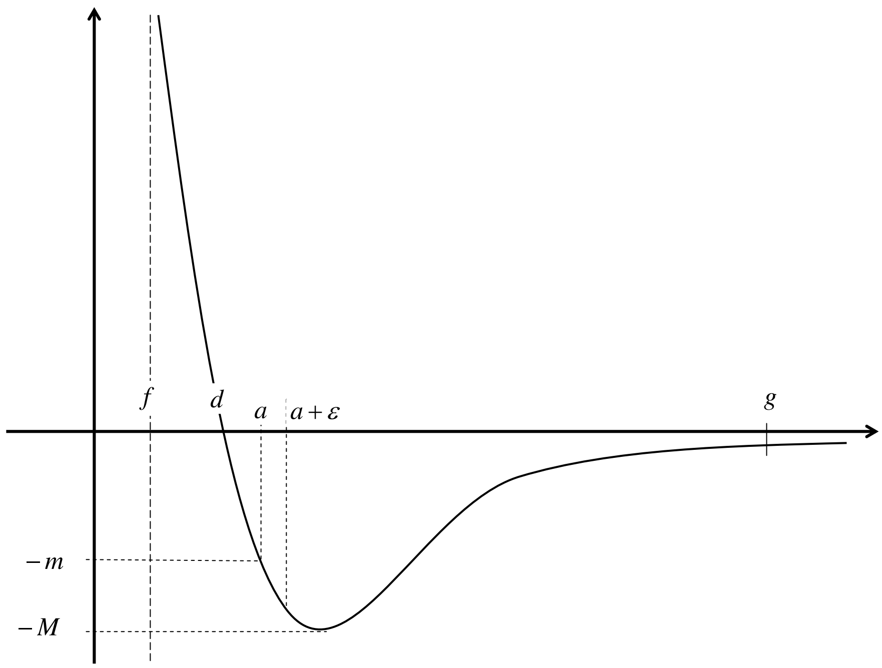

Figure 2: The potential function.

The construction of the event $\mathbf{\Omega^1}(\mathcal{R})$

Let m be a positive number such that $m < M$. For such m there exist positive numbers a and ε such that $\varepsilon \leq \delta$ and

$$\varphi(x) \leq -m \text{ for all } |x| \in [a, a+\varepsilon],$$

(see Figure 2.)

Let $\gamma = \bigcup_{\langle c,c' \rangle \in \Gamma(\mathcal{R})} \langle c, c' \rangle$ be the line in $\mathbb{R}^2$ composed of the bonds $\langle c, c' \rangle \in \Gamma(\mathcal{R})$. The length of γ is equal to $(2d+\delta)n(\mathcal{R})$. We introduce a direction on γ. Let it be counterclockwise. The points of any finite subset $\widetilde{D}$ of γ we numerate along the direction of γ, that is $\widetilde{D} = \{x_0, ..., x_k\}$. The point $x_{i+1} \in \widetilde{D}$ is the first point which can be reached from x_i in the direction of γ.

Let $B_{a+\frac{\varepsilon}{2}}(x_i)$ be the closed disc of the radius $a + \frac{\varepsilon}{2}$ with its center at $x_i \in \widetilde{D}$. If $a + \frac{\varepsilon}{2} < 4d + 2\delta$ then $B_{a+\frac{\varepsilon}{2}}(x_i) \cap \gamma$ is connected. In the case

$a + \frac{\varepsilon}{2} \geq 4d + 2\delta$ the set $B_{a+\frac{\varepsilon}{2}}(x_i) \cap \gamma$ can be unconnected. Let $C(x_i)$ be a connected component of $B_{a+\frac{\varepsilon}{2}}(x_i) \cap \gamma$ containing x_i.

Now we consider $\widetilde{D}$ only such that $x_{i+1} \in C(x_i)$ and $|x_{i+1} - x_i| = a + \frac{\varepsilon}{2}$ for any $i < k$. Every pair $x, y \in \widetilde{D}$ such that $|x-y| = a+\frac{\varepsilon}{2}$ we call *connected.* The number $\eta(\widetilde{D})$ of the connected pairs in $\widetilde{D}$ takes one of two values

$$\#(\widetilde{D}) - 1 \leq \eta(\widetilde{D}) \leq \#(\widetilde{D}).$$

The number $\#(\widetilde{D})$ of points in any $\widetilde{D}$ is not greater than $\left[\frac{(2d+\delta)n(\mathcal{R})}{a+\frac{\varepsilon}{2}}\right]$.

Let $D = D(\mathcal{R})$ be such that $\#(D) = \sup_{\widetilde{D}} \#(\widetilde{D})$.

An inverse estimate is in the next

Lemma 3.4. *There exists $\alpha > 0$ such that for any contour $\mathcal{R}$ the number*

$$\#(D(\mathcal{R})) \geq \alpha n(\mathcal{R}) \tag{3.26}$$

if $n(\mathcal{R}) > \frac{2\sqrt{2}a}{d}$.

Proof We consider two different cases distinguished by the following inequalities

case 1. $a + \frac{\varepsilon}{2} < 4d + 2\delta$,

case 2. $a + \frac{\varepsilon}{2} \geq 4d + 2\delta$

The length $s(x_i, x_{i+1})$ of the piece of $C(x_i) \subseteq \gamma$ between x_i and x_{i+1} can be estimated as following

$$s(x_i, x_{i+1}) \leq \begin{cases} \sqrt{2}\left(a + \frac{\varepsilon}{2}\right) & \text{in the case 1.,} \\ \left(a + \frac{\varepsilon}{2}\right)\left(\frac{a+\frac{\varepsilon}{2}}{2d+\delta} + 1\right) & \text{in the case 2.} \end{cases}$$

Since

$$\#(D) \geq \frac{n(\mathcal{R})\left(a + \frac{\varepsilon}{2}\right)}{s(x_i, x_{i+1})}$$

we can take

$$\alpha \geq \begin{cases} \frac{1}{\sqrt{2}} & \text{in the case 1.} \\ \frac{2d+\delta}{a+\frac{\varepsilon}{2}+2d+\delta} & \text{in the case 2.} \end{cases}$$

$\square$

Let $B_{\frac{\varepsilon}{4}}(x)$ be the disc of the radius $\frac{\varepsilon}{4}$ centered at $x \in D$ and $U = \cup_{x\in D} B_{\frac{\varepsilon}{4}}(x)$. Every disc $B_{\frac{\varepsilon}{4}}(x)$ is called a *bead* and the set U is *necklace*. The set

$$\Sigma_U = \{\sigma \in \Omega_U : \forall x \in D, \ \#(\sigma \cap B_{\frac{\varepsilon}{4}}(x)) = 1, \sigma \cap U^c = \varnothing\}$$

is a set of configurations all particles of which are located in the beads only, one particle in every bead.

The configuration set $\Omega^1(\mathcal{R})$ contains configurations composed by the joint of three configurations:

$$\Omega^1(\mathcal{R}) = \{\omega_1 = \sigma \vee \phi_{W(\mathcal{R})\setminus U} \vee \widetilde{\omega} : \ \sigma \in \Sigma_U, \widetilde{\omega} \in \widetilde{\Omega}, \} \tag{3.27}$$

where $\phi_{W(\mathcal{R})\setminus U}$ is the empty configuration in $W(\mathcal{R}) \setminus U$ and $\widetilde{\omega} \in \widetilde{\Omega}$ are configurations in $W(\mathcal{R})^c$.

The lower bound for $\mathbf{P}^{\beta,\lambda}(\mathbf{\Omega^1}(\mathcal{R}))$

The probability of $\Omega^1(\mathcal{R})$ is

$$P_V(\Omega^1(\mathcal{R})) = \frac{1}{Z_V} \int_{\Omega^1} e^{-\beta H_V(\sigma)} e^{-\beta H_V(\widetilde{\omega})} e^{-\beta F(\sigma,\widetilde{\omega})} \Pi(\mathrm{d}(\sigma \vee \phi_{W(\mathcal{R})\setminus U} \vee \widetilde{\omega})).$$

The energy $F(\sigma, \tilde{\omega})$ of the interaction of σ and $\widetilde{\omega}$ is negative because the distance between any particles of $\widetilde{\omega}$ and of σ is greater than d, hence $e^{-\beta F(\sigma,\widetilde{\omega})}$ is greater than 1. Since $\phi_{W(\mathcal{R})\setminus U} = \varnothing$, then

$$P_V(\Omega^1(\mathcal{R})) \geq \frac{e^{-\lambda n \Delta} e^{\lambda \frac{\pi\varepsilon^2}{16} \#(D)}}{Z_V} \int_{\Sigma_U} e^{-\beta H_V(\sigma)} \Pi(\mathrm{d}\sigma) \int_{\widetilde{\Omega}} e^{-\beta H_V(\widetilde{\omega})} \Pi(\mathrm{d}\widetilde{\omega}). \tag{3.28}$$

To estimate $H_V(\sigma)$ remark that there exist at least $\#(D)-1$ the connected pairs in D. The interaction energy of any connected pair $x, y \in D$ is estimated from below as

$$\varphi(x-y) \leq -m.$$

Other pairs $x, y \in D$ which are not connected, that is $|x-y| \neq a+\frac{\varepsilon}{2}$, interact with a negative energy.

Hence

$$\int_{\Sigma_U} e^{-\beta H_V(\sigma)} \Pi(\mathrm{d}(\sigma)) \geq e^{m\beta(\#(D)-1)} \left(\frac{\lambda\pi\varepsilon^2}{16}\right)^{\#(D)} e^{-\frac{\lambda\pi\varepsilon^2}{16}\#(D)}$$

and it follows from (3.28) that

$$P_V(\Omega^1(\mathcal{R})) \geq e^{m\beta(\#(D)-1)} \left(\frac{\lambda\pi\varepsilon^2}{16}\right)^{\#(D)} P_V(\Omega^0(\mathcal{R})).$$

Defining $c(\beta) = e^{\beta m}$ and

$$G(\beta, \lambda) = \beta m + \ln \lambda + \ln\left(\frac{\pi\varepsilon^2}{16}\right) \tag{3.29}$$

we obtain

$$\begin{aligned} P_V(\Omega^0(\mathcal{R})) &\leq \exp\{-\#(D)G(\beta, \lambda)\} \exp\{\beta m\} \\ &= c(\beta) \exp\{-n(\mathcal{R})\alpha G(\beta, \lambda)\}. \end{aligned} \tag{3.30}$$

The inequality (3.24) holds in the infinite volume since the right hand side of (3.30) does not depend on V.

Taking

$$\beta^+_{\ell,h}(\lambda) = -\frac{1}{m}\ln(\lambda) - \frac{1}{m}\ln\left(\frac{\pi\varepsilon^2}{16}\right) + \frac{h}{m}$$

we complete the proof of Lemma 3.3. □

Next we define

$$\beta^+_\ell(\lambda) = \beta^+_{\ell, \frac{\ln(c)}{\alpha}}(\lambda), \tag{3.31}$$

where c is a combinatorial constant such that the number of the contours of the length n surrounding $0 \in \mathbb{R}^2$ is not greater than c^n. It is known that $c \leq 3$. Let λ^+_ℓ be the solution of the equation $\beta^+_\ell(\lambda) = 0$. Define the set

$$A^+ = \{(\beta, \lambda) : \ \lambda \leq \lambda^0, \ \beta > \beta^+_\ell(\lambda)\} \cup \{(\beta, \lambda) : \ \lambda > \lambda^+_\ell, \ \beta \geq 0\}$$

Lemma 3.5. *If $(\beta, \lambda) \in A^+$ then with the probability 1 there exists only a finite number of c-contours surrounding $0 \in \mathbb{R}^2$.*

Proof Let $\Omega^0(\mathcal{R})$ be the set of all configurations containing a c-contour $\mathcal{R}$, the empty contour which surrounds $0 \in \mathbb{R}^2$, and let $\Omega^0_n = \bigcup_{\mathcal{R}:\, n(\mathcal{R})=n} \Omega^0(\mathcal{R})$. Then $\Omega^0 = \bigcup_n \Omega^0_n$ and

$$\sum_{n\geq 1} P^{\beta,\lambda}(\Omega^0_n) \leq \exp\{\beta m\} \sum_{n\geq 1} \exp\left\{-n\Big(\alpha G(\beta(\lambda), \lambda) - \ln(c)\Big)\right\} < \infty \tag{3.32}$$

if $(\beta, \lambda) \in A^+$. It follows from (3.32) that

$$P^{\beta,\lambda}\left(\bigcap_m \bigcup_{n=m}^{\infty} \Omega_n^0\right) = 0. \tag{3.33}$$

The inequality (3.33) means that with the probability 1 there exists a finite number of the empty contours surrounding 0. $\square$

Let ω be a configuration. The set $Q_\omega = \bigcup_{x \in \omega} B_{\ell/2}(x)$ can be represented as the union of $\ell/2$-neighborhoods of ℓ-clusters which are connected components.

We define now a *b-contour* (Boolean contour). Assume that there exists a line $L \subseteq Q_\omega^c$ surrounding $0 \in \mathbb{R}^2$ such that $K_L \cap \omega = \varnothing$, where

$$K_L = \bigcup_{x \in L} B_\ell(x).$$

The set K_L is called a *b-contour* surrounding 0 or simply a *b-contour*.

The $\ell/2$ neighborhood of any ℓ-cluster does not intersect L.

Lemma 3.6. *Assume that $\mathbb{R}^2$ is split into cells of the linear size q. For any b-contour K with a radius r, $r > \sqrt{2}q$, there exists a c-contour $\mathcal{R}$ such that $W(\mathcal{R}) \subseteq K$.*

Proof The proof is based on the following simple observation: if we cast a coin of the radius r on the plane $\mathbb{R}^2$ divided into the square cells $\mathcal{S}$, then there exists a cell which will be covered entirely by the coin. Moreover if the center of the coin lies on a boundary of two cells or four cells (one point) then all those cells are covered by the coin. $\square$

We say that two b-contours are different if c-contours included into them are different. Since the number of the c-contours is finite with the probability 1 then the number of different b-contours is finite as well. Therefore there exists an infinite component in Q_ω for almost all ω. $\square$

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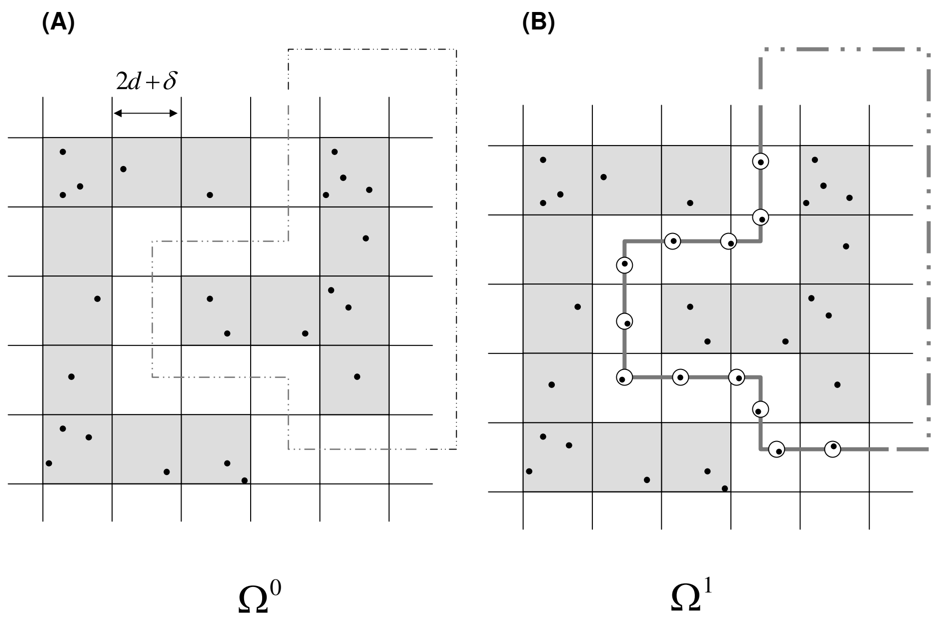

Figure 3: The region of the positivity of the percolation function.

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